Fourier Transform

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2dx Workshop
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Jean Baptiste Joseph Fourier

France, 1768 - 1830

$$F(u) = \frac{1}{2\pi} \int f(x) \cdot e^{-i \cdot 2\pi \cdot u \cdot x} \, dx$$

$$PS(u) = |F(u)|^2$$

Jean Baptiste Joseph Fourier

France, 1768 - 1830

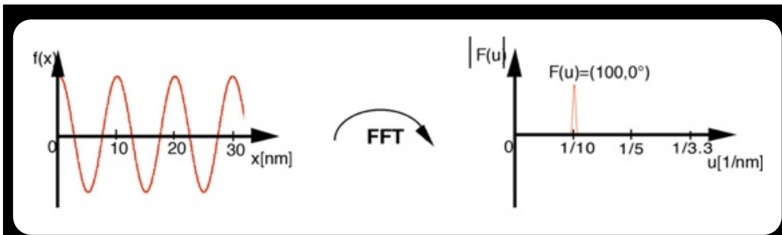
$$F(u)=\frac{1}{2\pi}\int f(x)\cdot e^{-i\cdot 2\pi\cdot u\cdot x}\,dx$$

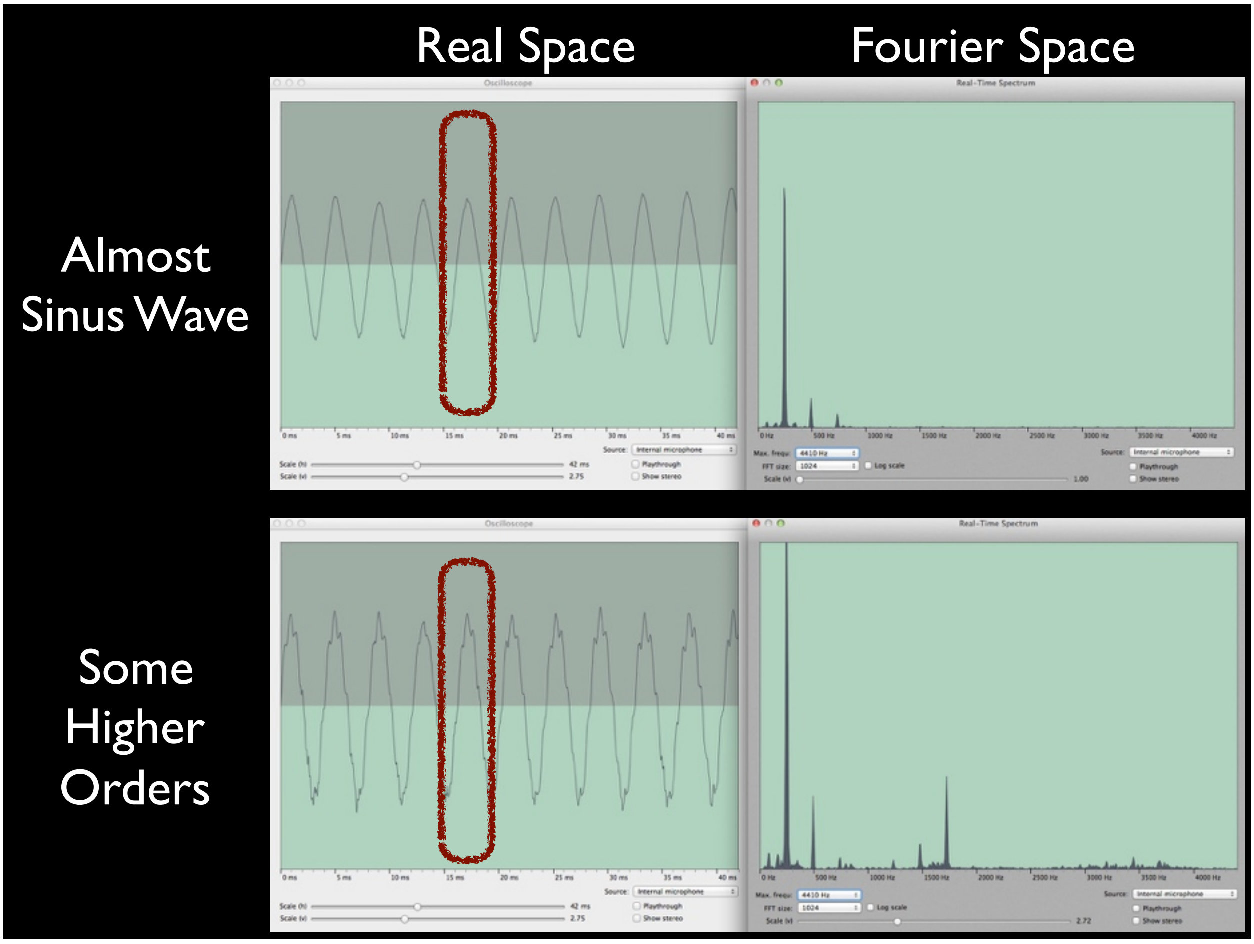
$$F(u)=\frac{1}{2\pi}\int f(x)\cdot\left[\cos(2\pi\cdot u\cdot x)-i\cdot\sin(2\pi\cdot u\cdot x)\right]dx$$

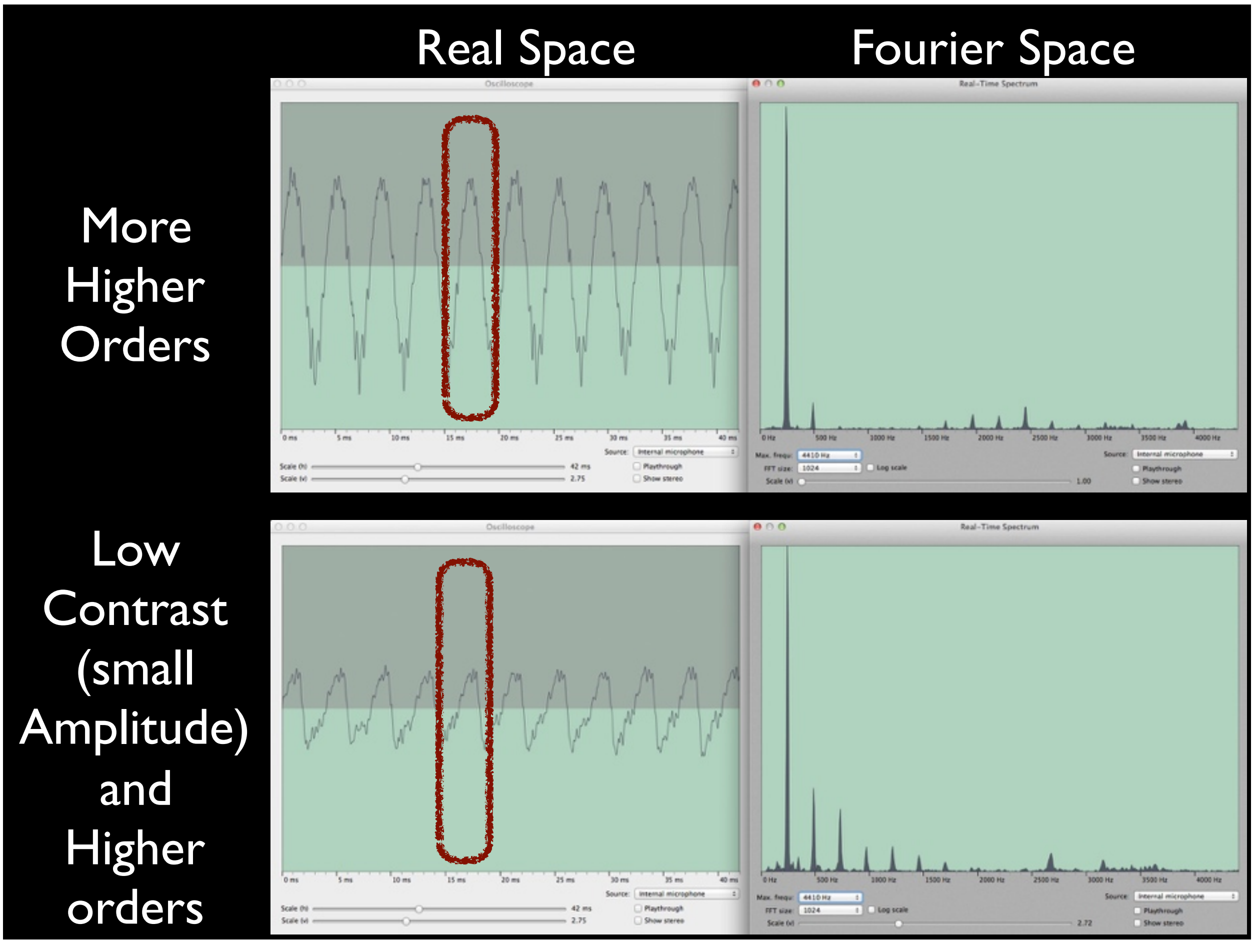
$$F(u)=FT\left(f(x)\right)$$

$$PS(u)=\left|F(u)\right|^2$$

1D FFT Examples



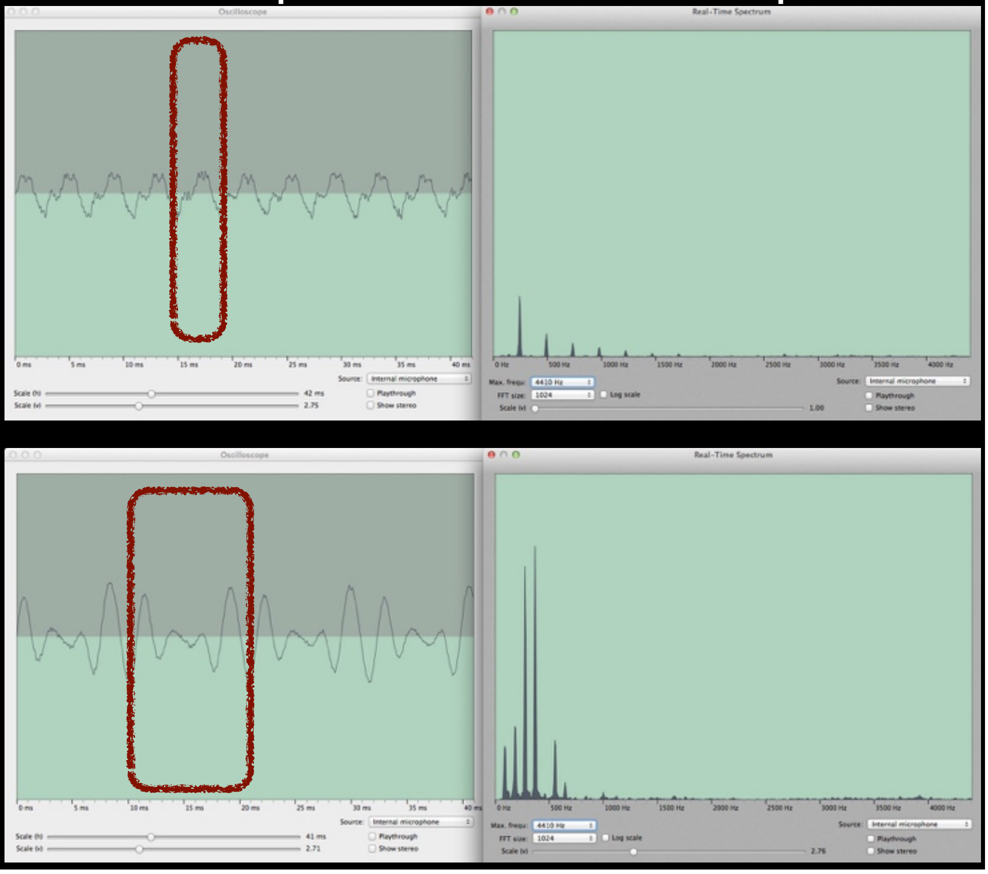

Real Space
Fourier Space
Almost Sinus Wave
Some Higher Orders

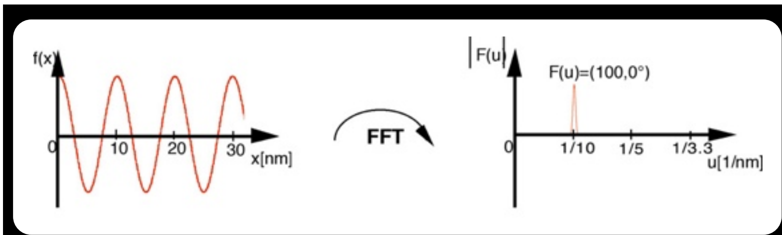

Real Space
Fourier Space
More Higher Orders
Low Contrast (small Amplitude) and Higher orders

Real Space | Fourier Space

Low Amplitude and higher orders

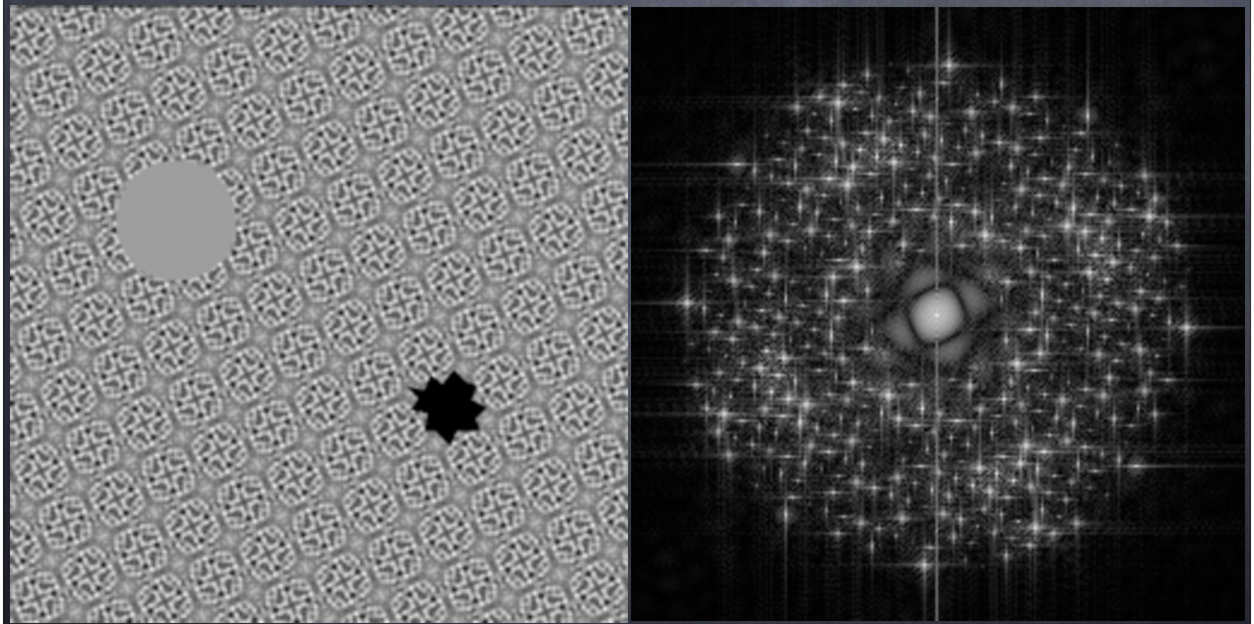
Large structure, or lower base frequency, and higher orders





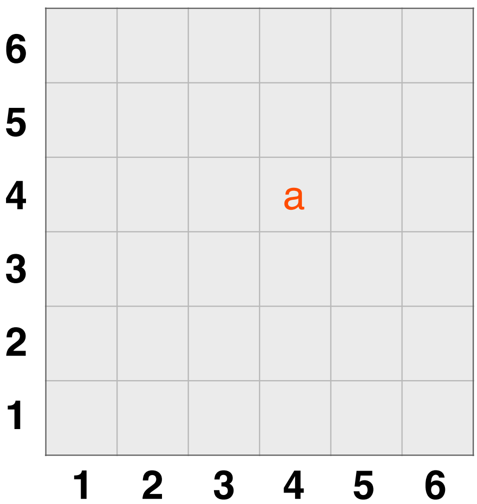
2D FFT Examples

Real Space | Fourier Transform

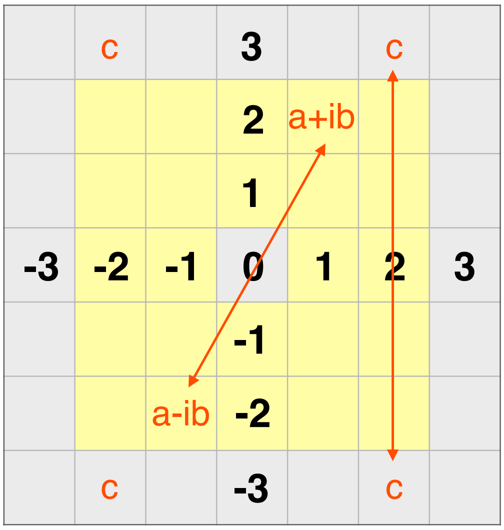


2D Fourier Transforms in the Computer Memory

Real-Space image, here of 6x6 pixels



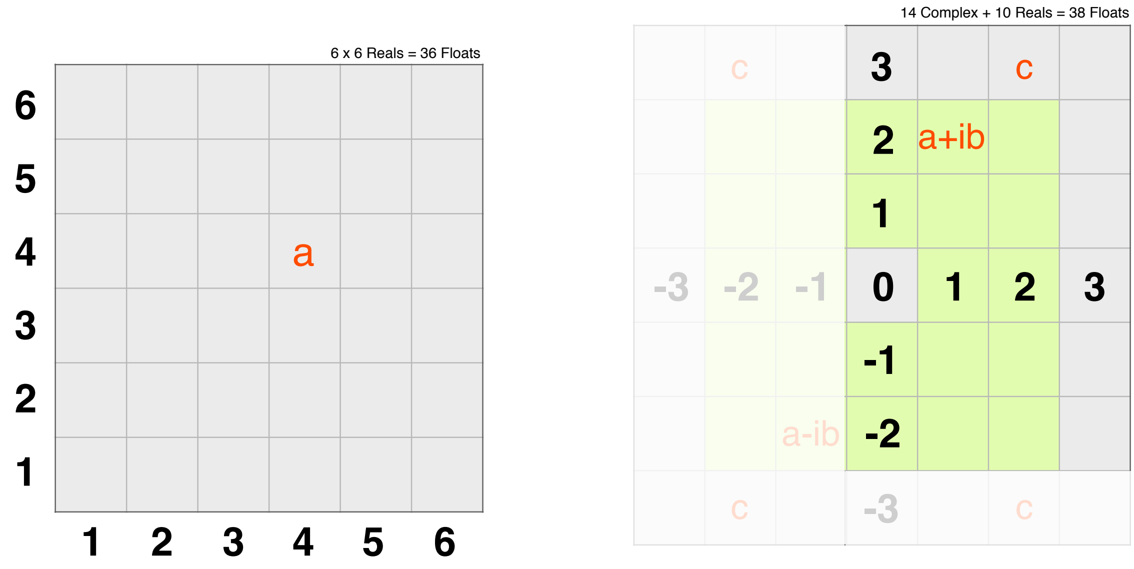
Fourier Transform of the same image



- The FT of real functions (e.g. images) are Hermitian: for every point (a+bi), there is a corresponding point (a-bi)
- For an N × N pixel image, Fourier transform is N/2+1 × N
- The positive Nyquist and negative Nyquist values are the same

(after John Rubinstein, NRAMM 2014)

2D Fourier Transforms in the Computer Memory



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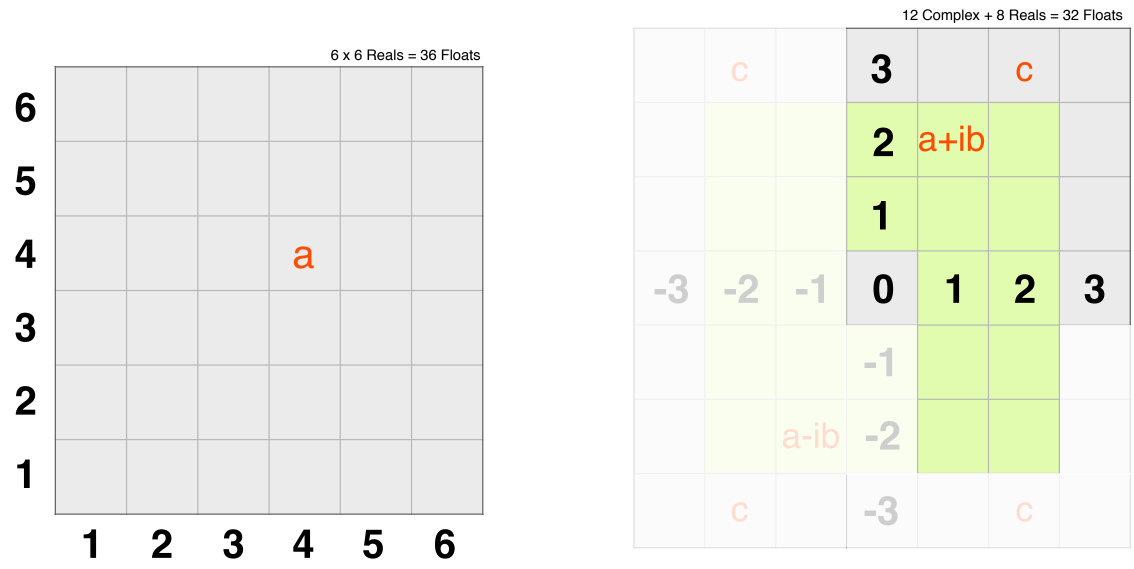

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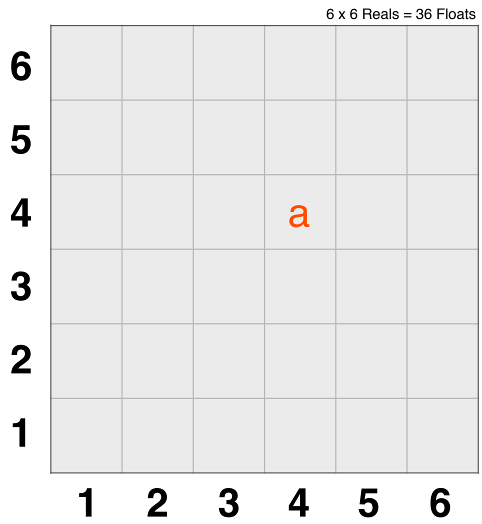
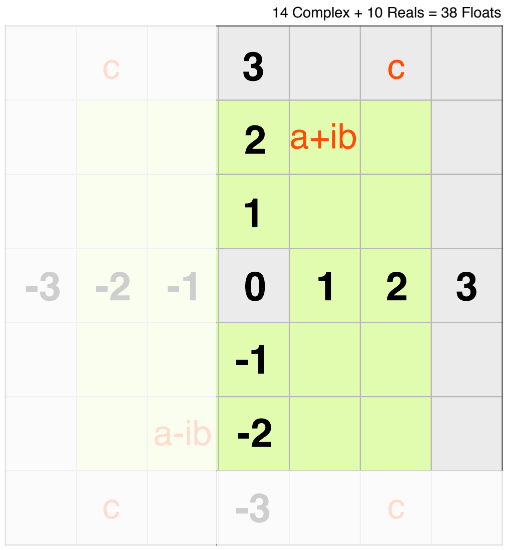
2D Fourier Transforms in the Computer Memory





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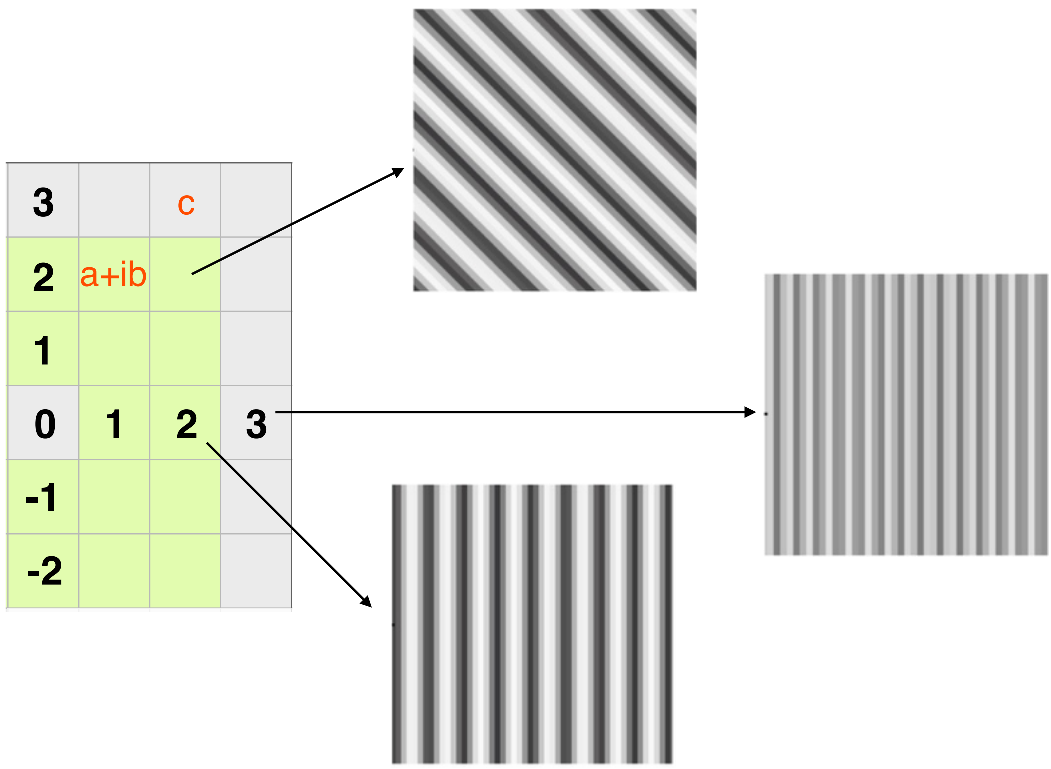
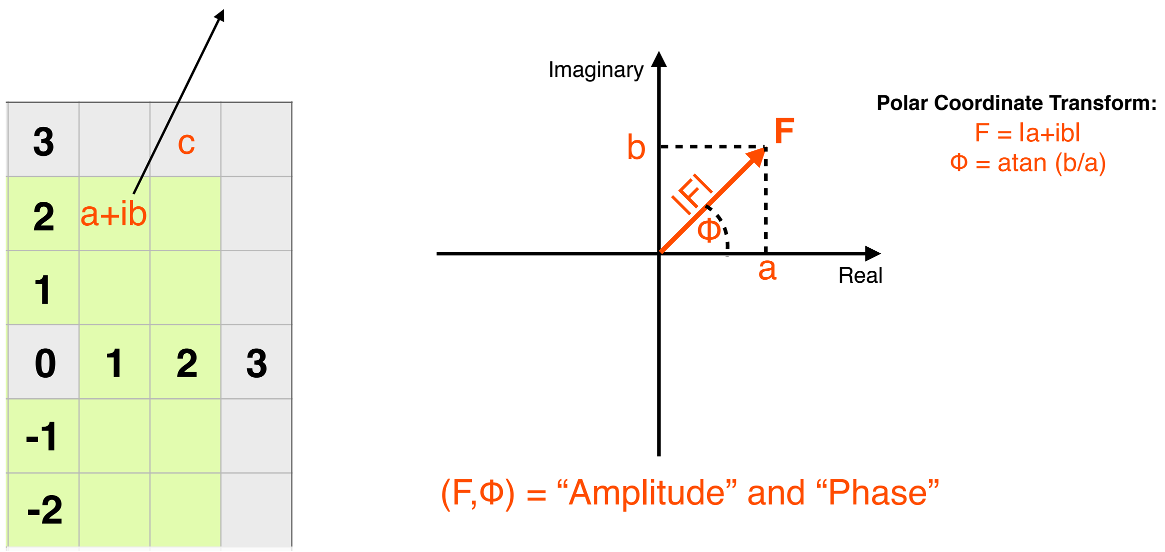
2D Fourier Transforms in the Computer Memory



(after John Rubinstein, NRAMM 2014)

2D Fourier Transforms in the Computer Memory

a+ib = “Cosinus-Term (Real)” and “Sinus-Term (Imaginary)”



(F,Φ) = “Amplitude” and “Phase”

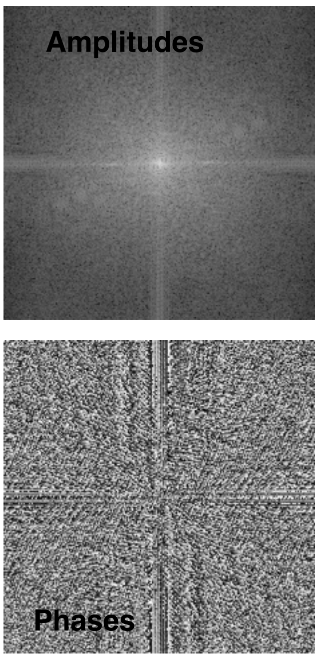
(after John Rubinstein, NRAMM 2014)

Basics of image processing: Fourier transforms of 2D images

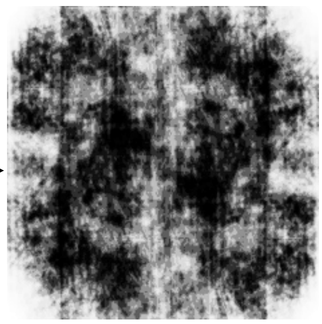
Fourier space

Real space

FT



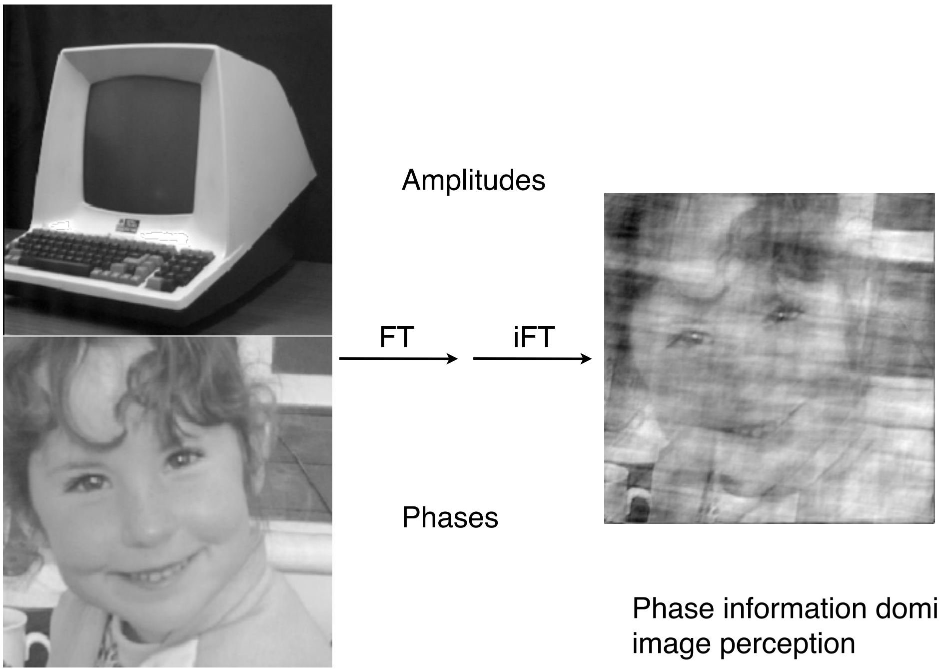
FT



Phase information dominates image perception

http://homepages.inf.ed.ac.uk/rbf/HIPR2/fourier.htm

Basics of image processing: Fourier transforms of 2D images

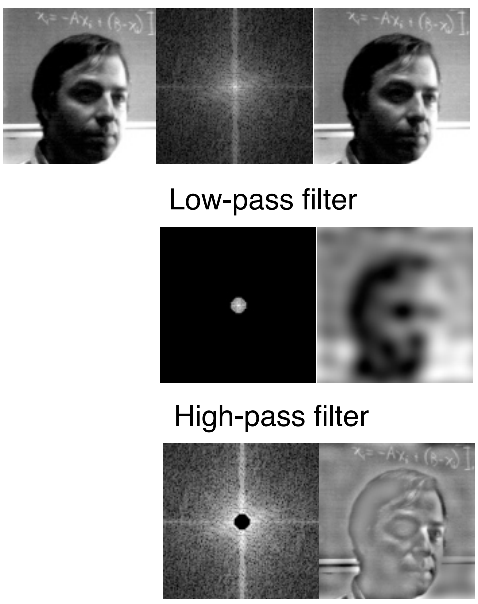


Phase information dominates image perception

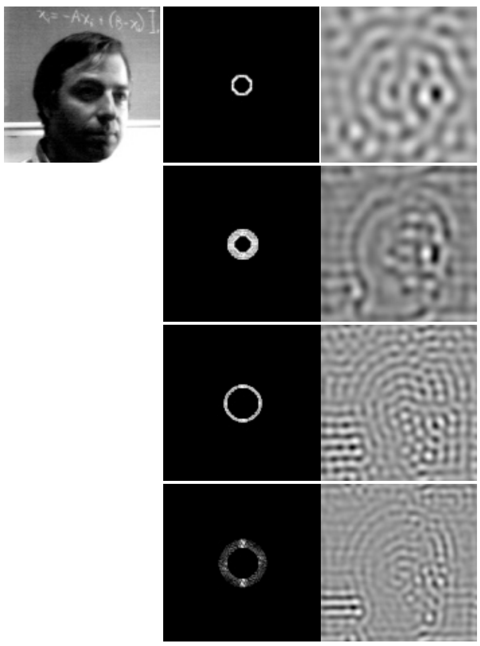
http://homepages.inf.ed.ac.uk/rbf/CVonline/LOCAL_COPIES/OWENS/LECT4/node2.html

Basics of image processing: Fourier filters of images

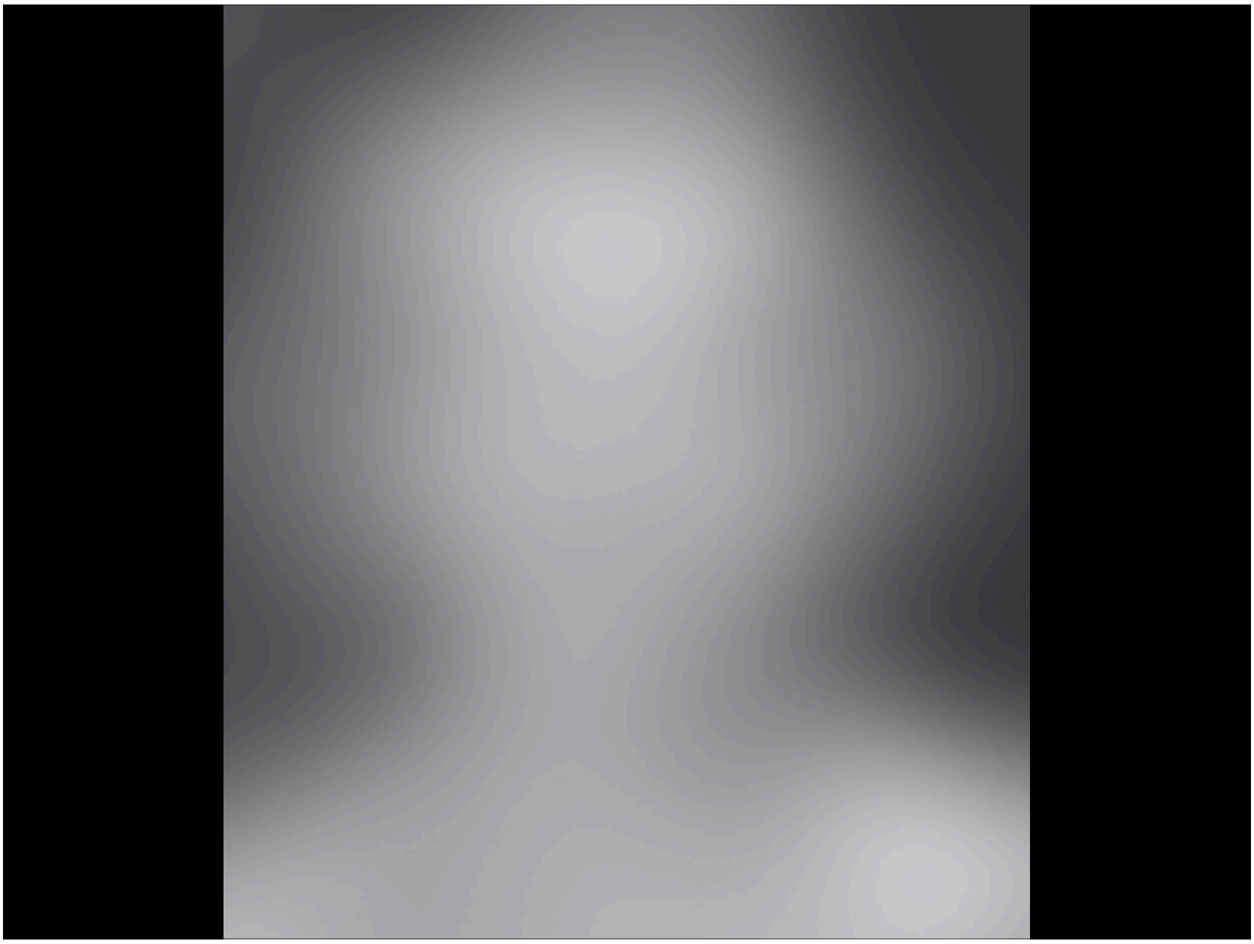
Real -->Fourier -->Real

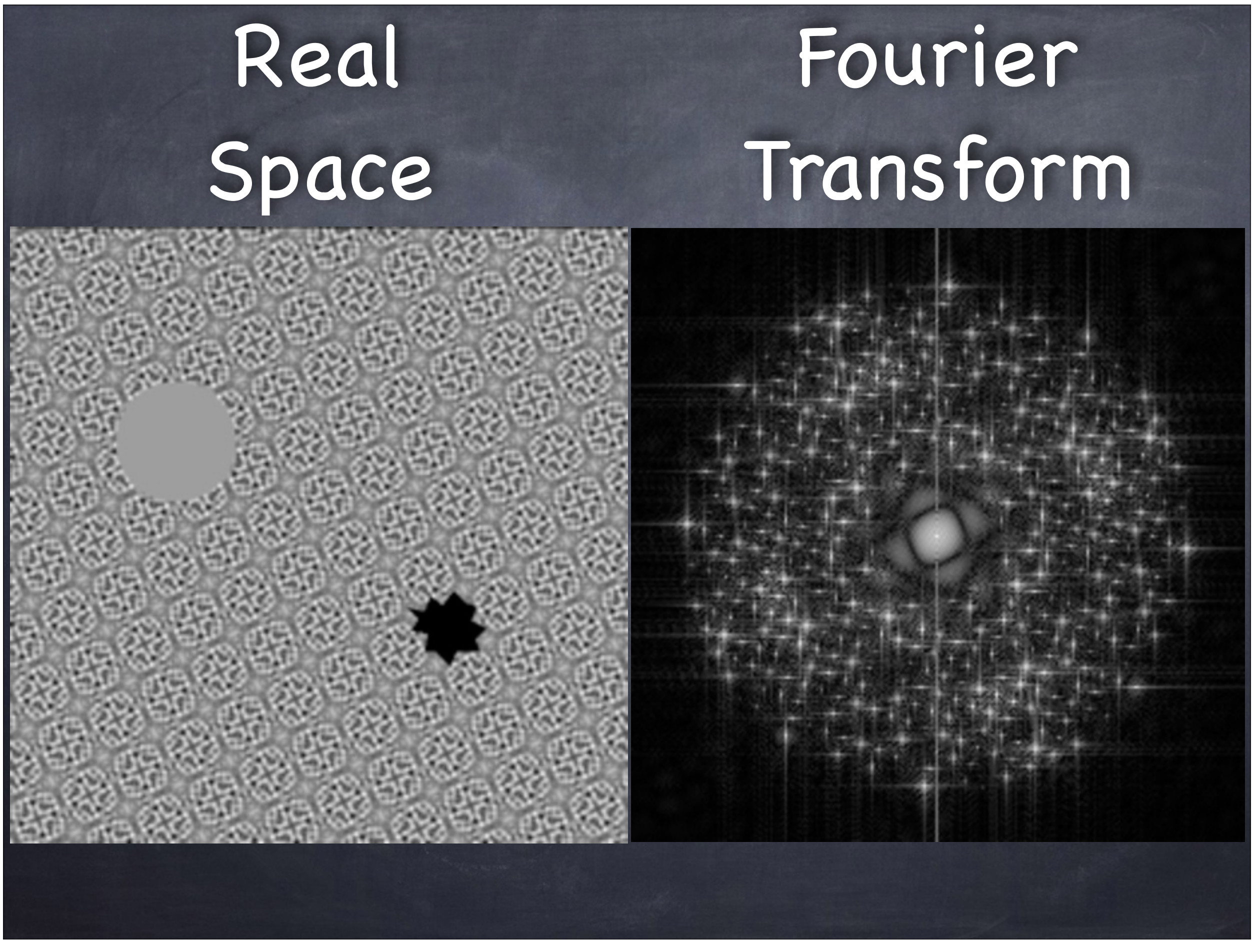


Band-pass

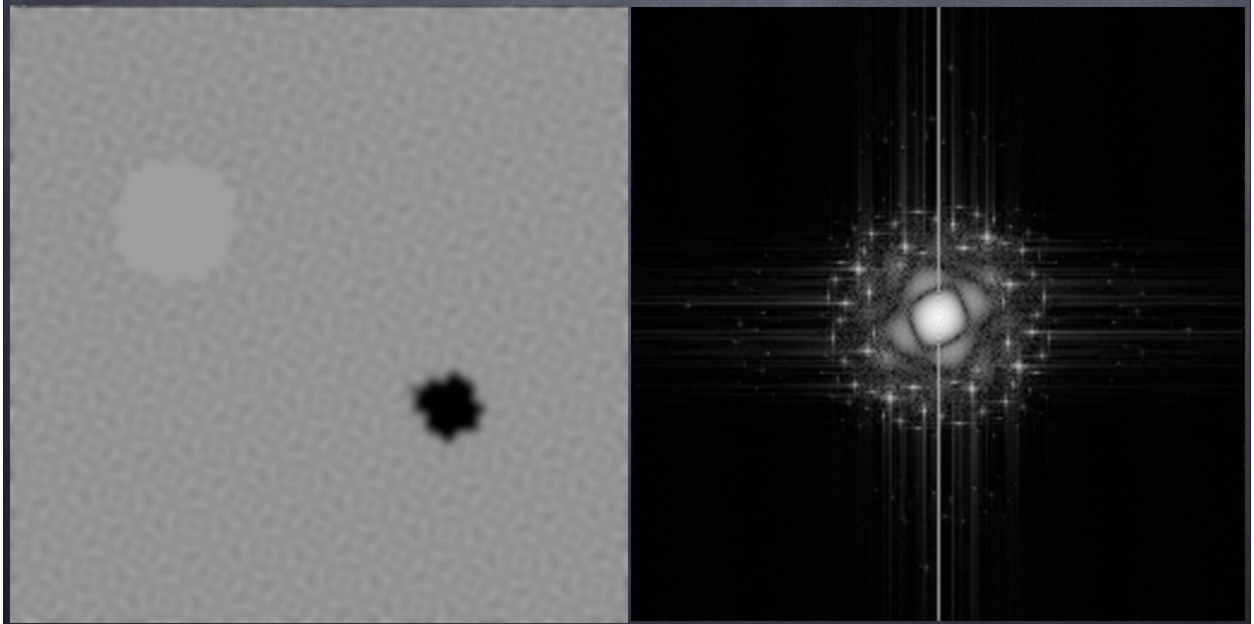


http://sharp.bu.edu/~slehar/fourier/fourier.html

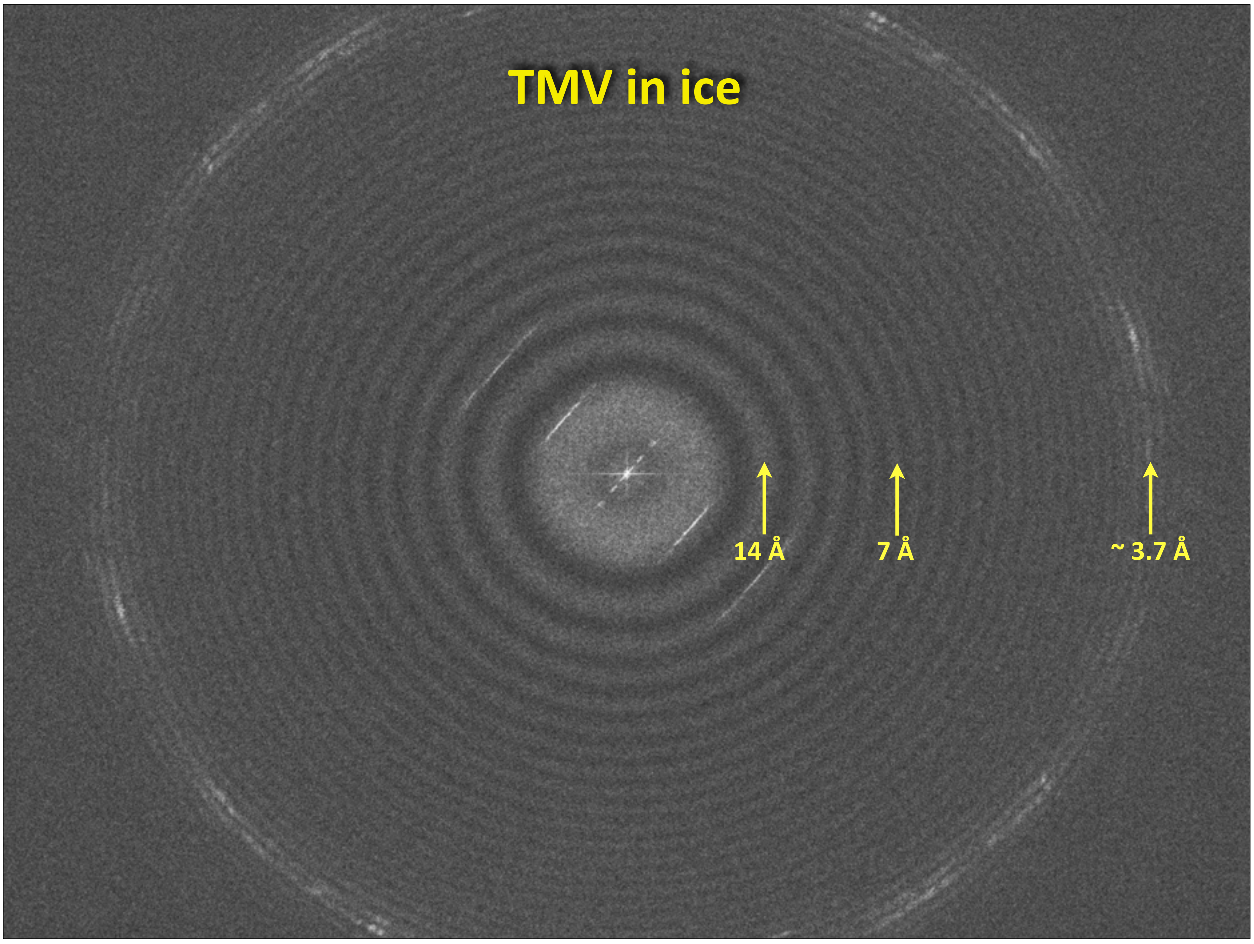


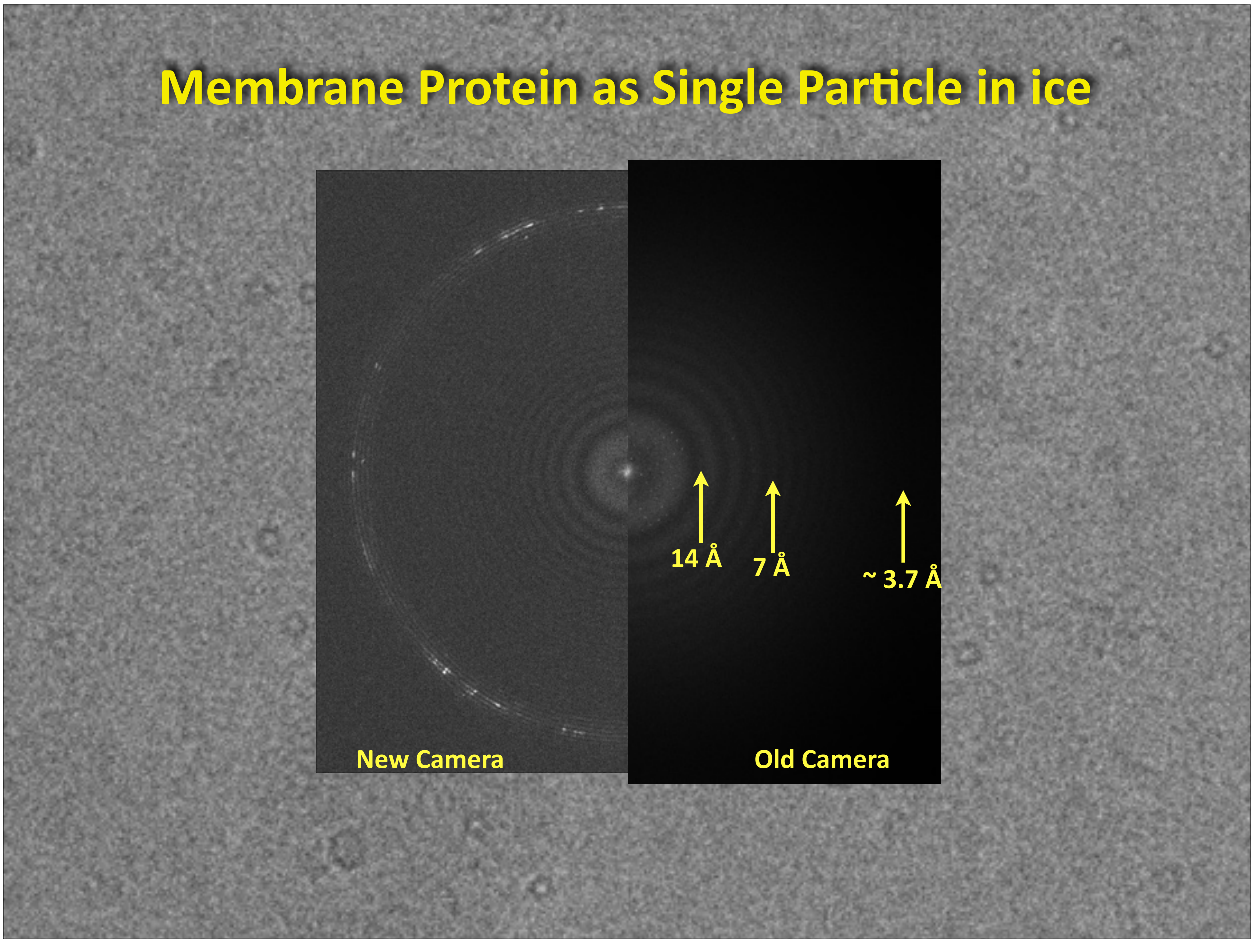

Real
Space
Fourier
Transform

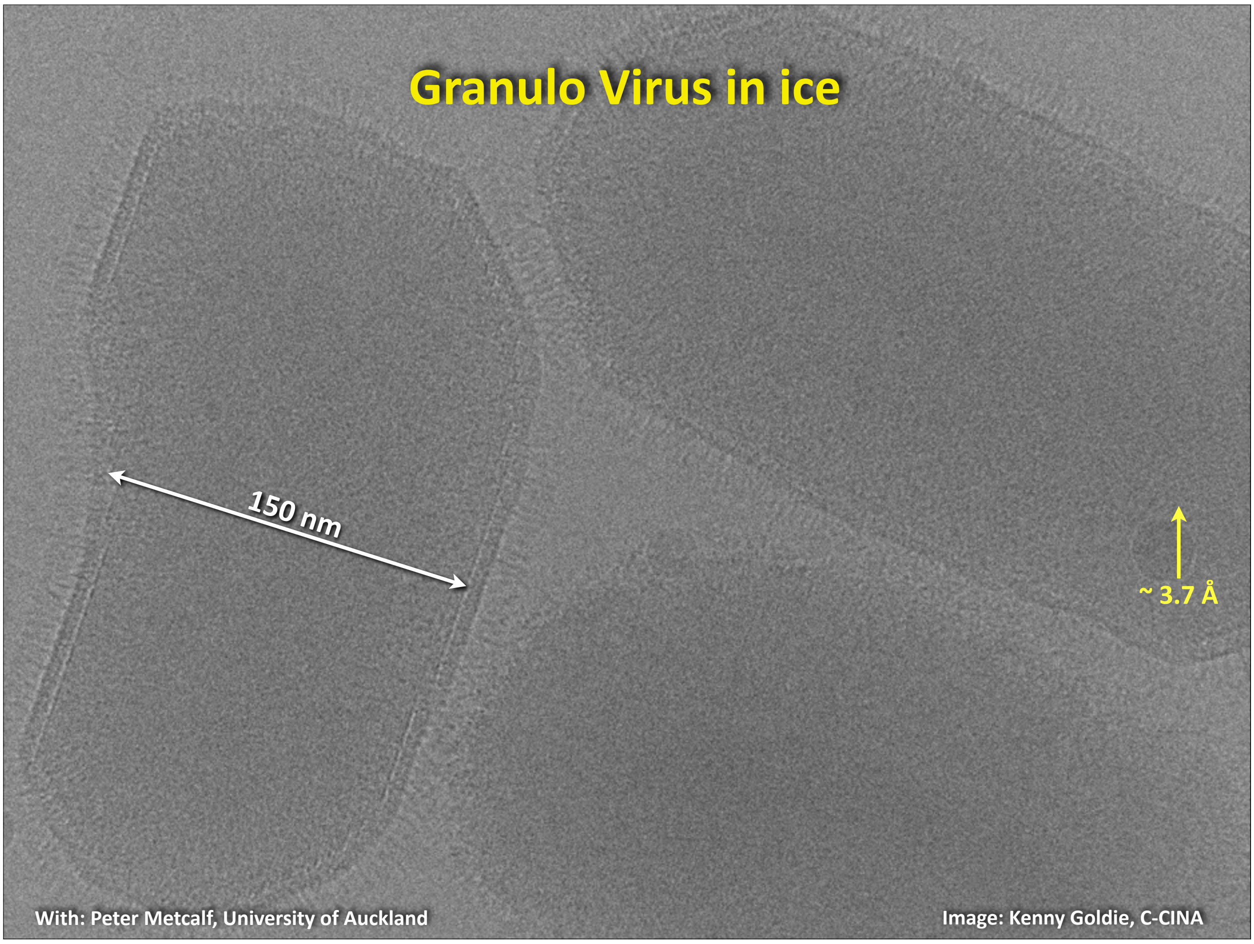
Real Space | Fourier Transform



Correct Envelope

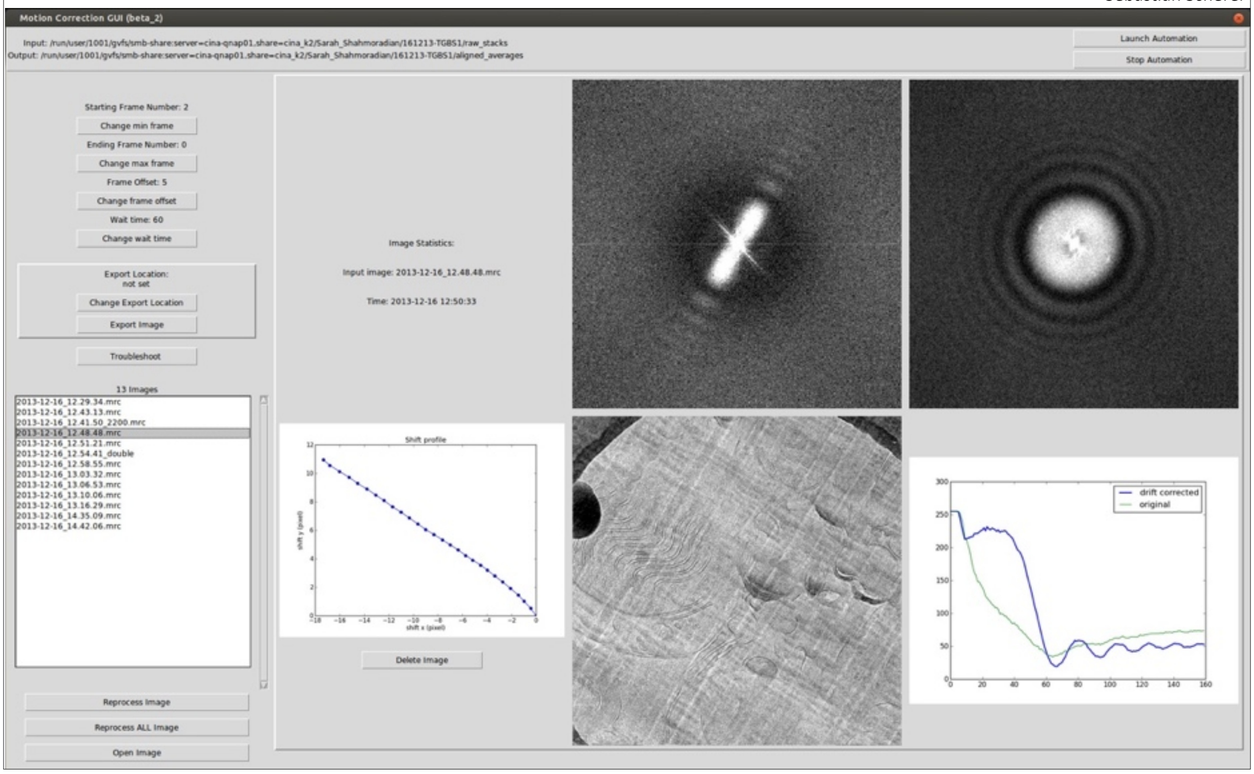

TMV in ice
14 Å
7 Å
~ 3.7 Å


Membrane Protein as Single Particle in ice
14 Å
7 Å
~ 3.7 Å
New Camera
Old Camera

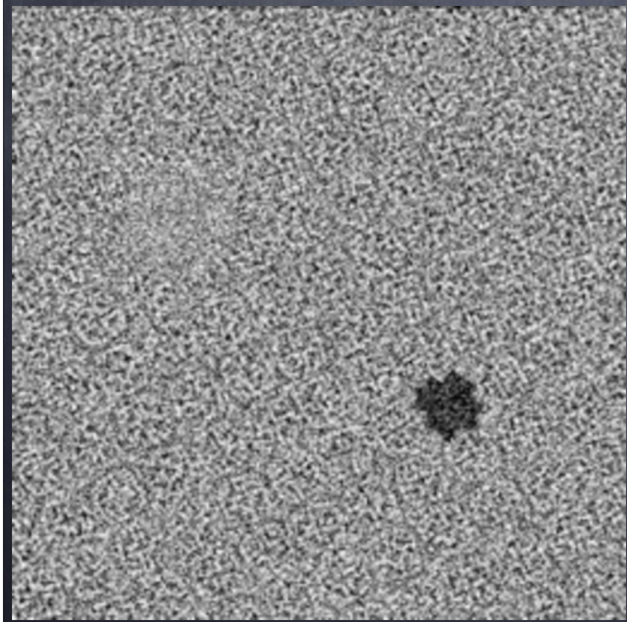


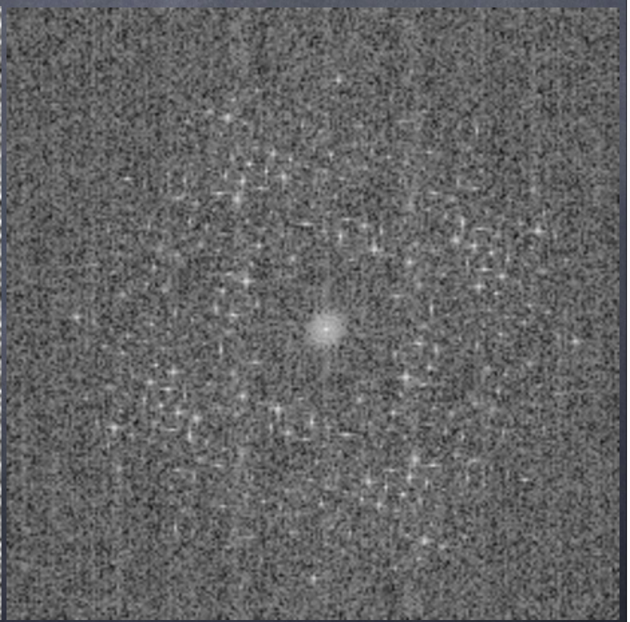
2dx_automator:
Automated movie frame alignment and averaging, based on MotionCorr(1) (Cheng lab) alignment tool.

Sebastian Scherer



Real Space | Fourier Transform



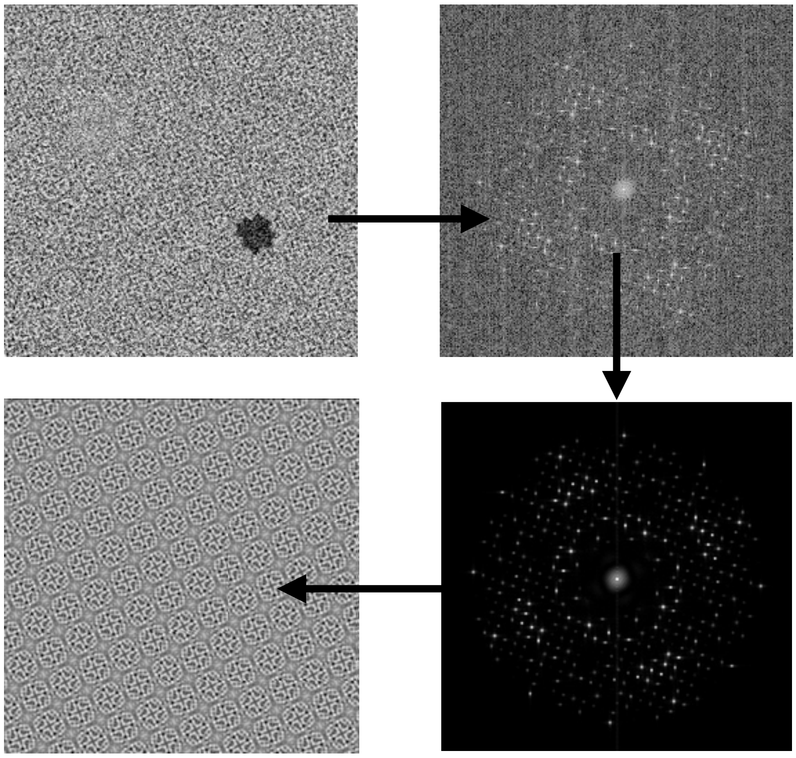


Filter Noise

Extract the information from the noise

Noisy image

FFT



Filtered image

Filtered FFT

Fourier Transformation

Fourier transformation.

$$F(u) = \frac{1}{2\pi} \int f(x) \cdot e^{-i \cdot 2\pi \cdot u \cdot x}\, dx$$

$$F(u) = \frac{1}{2\pi} \int f(x) \cdot \left[\cos(2\pi \cdot u \cdot x) - i \cdot \sin(2\pi \cdot u \cdot x) \right] dx$$

$$F(u) = FT\left(f(x)\right)$$

Inverse Fourier transformation exists.

$$f(x) = FT^{-1}\left(F(u)\right)$$

$$f(x) = \frac{1}{2\pi} \int F(u) \cdot e^{+i \cdot 2\pi \cdot u \cdot x}\, du$$

Fourier Equations

$$FT(a \cdot f(x)) = a \cdot F(u)$$

If you put more contrast in the image, then the FFT's amplitude gets stronger.

$$FT(f(x) + g(x)) = F(u) + G(u)$$

Adding two images f and g and calculating their FFT is like adding the FFTs F and G of them.

$$FT(f(ax)) = F(u / a)$$

If you stretch an image by a, then you shorten the FFT by a. (===> reciprocity)

$$FT(rotated\ \ f(x)) = rotated\ \ F(u)$$

If you rotate and image, then you also rotate its FFT.

Convolution

$$f(x) \otimes g(x) = FT^{-1}\left[F(u) \cdot G(u)\right]$$

Convolution of f with g in real space is slow. It can be done much faster by multiplying their FFTs, and calculating the inverse FFT of the result.

"Convolution of a set of spots with a duck produces a set of ducks."

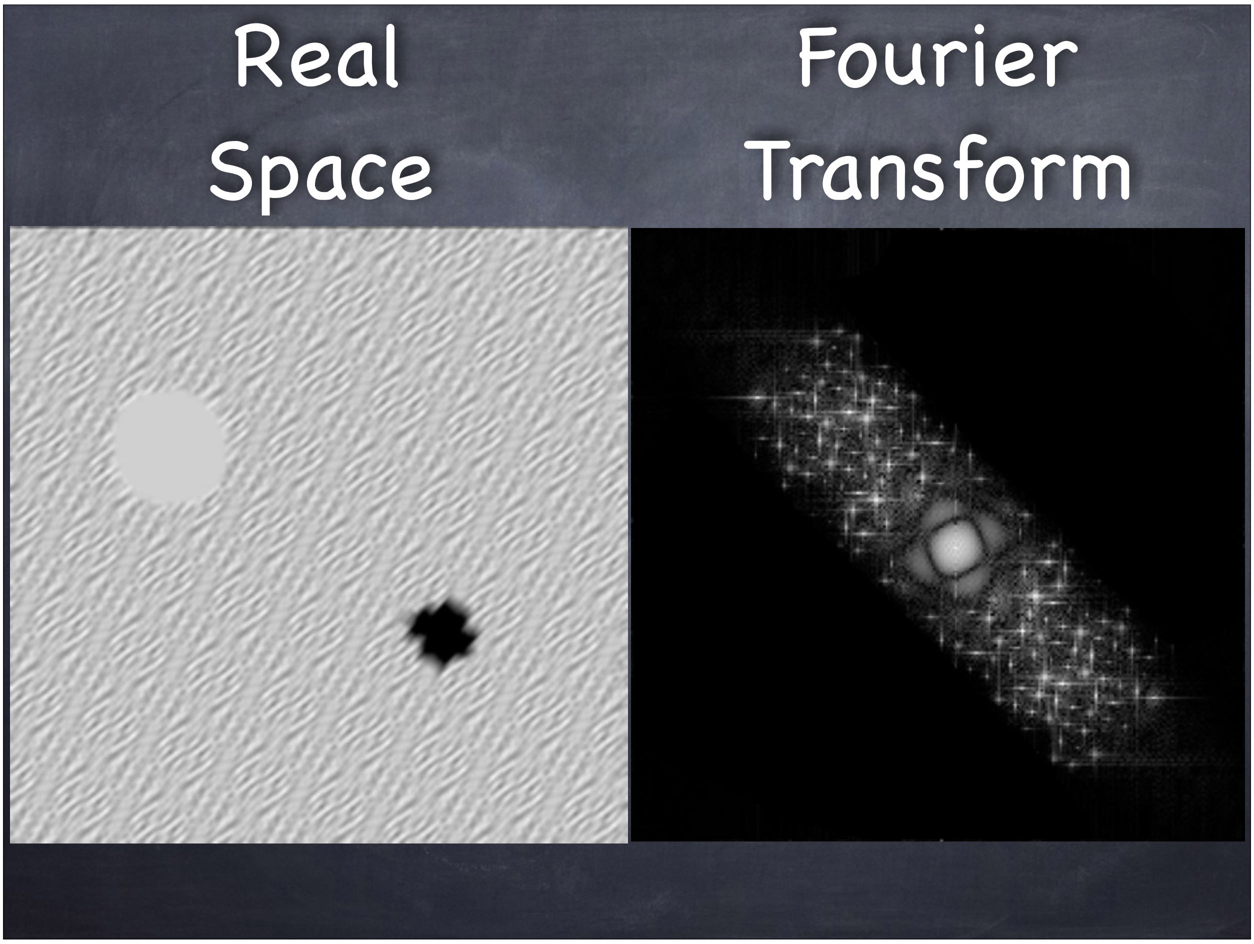
*"Convoluting a structure's projection map with a **Point Spread Function** gives you the recorded image. This is equivalent to multiplying the FFT of the map with the **Contrast Transfer Function**."*
(=> deconvolution)

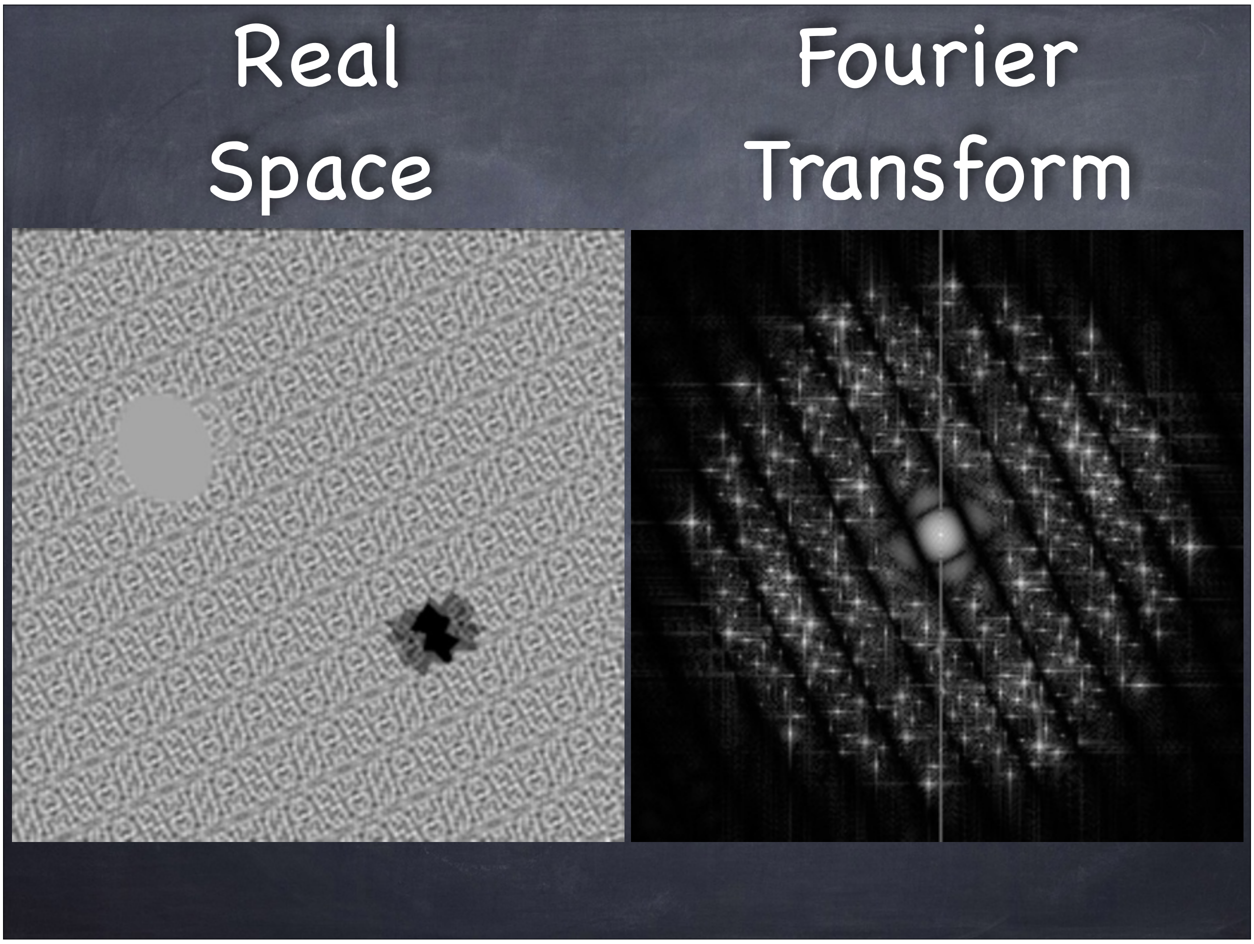
Cross-Correlation

$$f(x) \times g(x) = FT^{-1}\left[F(u) \cdot G^{*}(u)\right]$$

Cross-correlation of f with g in real space is slow. It can be done much faster by calculating their FFTs F and G, taking the complex conjugate of G, multiplying F with G*, and calculating the inverse FFT of the result.*

"Cross-correlation of a noisy image of many viruses with a virus-like circular reference produces a map with peaks that show where the viruses are."


Real
Space
Fourier
Transform


Real
Space
Fourier
Transform

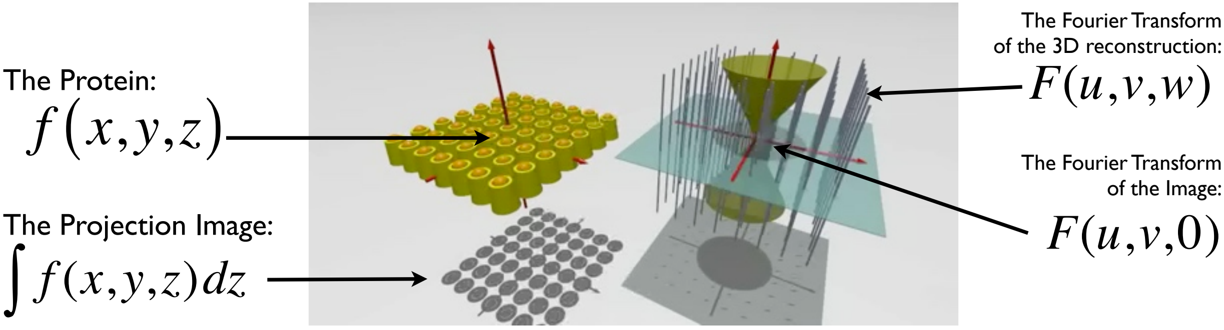
Central Section Theorem

$$F(u,v,w) = FT(f(x,y,z))$$

A 3D space with x,y,z corresponds to a Fourier space with u,v,w.

$$F(u,v,0) = FT\left[\int f(x,y,z)dz\right]$$

A projection in the vertical direction dz corresponds to the central section in the w=0 plane.



Special Cases

$$FT(rect(ax)) = \frac{1}{|a|} sinc\left(\frac{u}{a}\right)$$

The Fourier transform of a rectangular function is the sinc function.

$$FT(sinc(ax)) = \frac{1}{|a|} rect\left(\frac{u}{a}\right)$$

The Fourier transform of a sinc function is a rectangular function.

$$FT(e^{-ax^2}) = \sqrt{\frac{\pi}{a}} \cdot e^{-\frac{(\pi u)^2}{a}}$$

The Fourier transform of a Gaussian function is a Gaussian function.

$$FT(\delta(x)) = 1$$

The Fourier transform of a delta function is a constant 1.

$$FT(1) = \delta(u)$$

The Fourier transform of a constant function is a delta function.

Fourier Transformation

$$F(u) = \frac{1}{2\pi} \int f(x) \cdot e^{-i \cdot 2\pi \cdot u \cdot x} \, dx$$

- *Analyze performance of microscope*
- *Analyze resolution and quality of image*
- *Modify or correct certain image artifacts (=> CTF)*

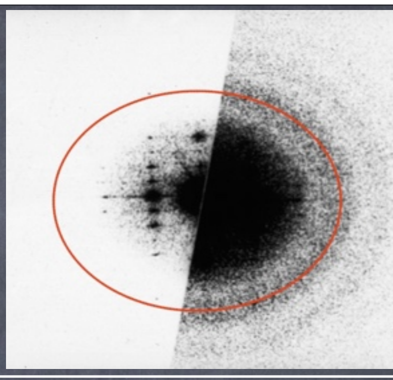
- *In case of crystal: Extract structure (=> Fourier filtering)*

- *Use to*
  Calculate cross-correlation function with a reference
  Calculate convolution with or deconvolution of a kernel function

- *In case of different sample orientations in set of images:*
  - *combine into 3D reconstruction (=> Backprojection)*

What the FFT can tell us

(After David deRosier, 2006)



Spot positions	Unit cell size and shape
Spot size	Size of coherent domains
Intensity relative to background	Signal to noise ratio
Distance to farthest spot	Resolution
Amplitude and Phase of spots	Structure of molecules
Radius of Thon rings	Amount of defocus
Ellipticity of Thon rings	Amount of astigmatism
Assymetric intensity of Thon rings	Amount of instability
Direction of assymetry	Direction of instability